4.1. DERIVADAS DE LAS FUNCIONES TRIGONOMETRICAS

Derivada de $y = Sen(x)$

La derivada de $y = Sen(x)$ se puede obtener como:

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{Sen(x+h) - Sen(x)}{h}\right)$$

Para calcular este límite se utilizan los siguientes conceptos previamente estudiados:

- $\lim_{h\to 0}\frac{Sen(h)}{h} = 1$
- $\lim_{h\to 0}\frac{Cos(h)-1}{h} = 0$
- $Sen(x+h) = Sen(x)Cos(h) + Cos(x)Sen(h)$

Por lo tanto desarrollando el límite se tiene:

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{Sen(x+h) - Sen(x)}{h}\right)$$ *Definición de derivada*

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{Sen(x)Cos(h) + Cos(x)Sen(h) - Sen(x)}{h}\right)$$ *Aplicando suma de arcos*

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{Sen(x)(Cos(h)-1) + Cos(x)Sen(h)}{h}\right)$$ *Factorizando el numerador*

$$\frac{dy}{dx} = \lim_{h\to 0}\left(Sen(x)\frac{Cos(h)-1}{h}\right) + \lim_{h\to 0}\left(Cos(x)\frac{Sen(h)}{h}\right)$$ *Suma de Limites*

$$\frac{dy}{dx} = Sen(x)\lim_{h\to 0}\left(\frac{Cos(h)-1}{h}\right) + Cos(x)\lim_{h\to 0}\left(\frac{Sen(h)}{h}\right)$$ *Sacando las constantes fuera del límite*

$$\frac{dy}{dx} = Sen(x)\times 0 + Cos(x)\times 1 = Cos(x)$$ *Por los límites conocidos*

De donde:

$$\boxed{\frac{d}{dx}Sen(x) = Cos(x)}$$

Si ***u*** es una función diferenciable de ***x***, es posible aplicar la regla de la cadena así:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

en donde $y = Sen\,u$ para obtener como resultado:

$$\boxed{\frac{d}{dx}\left(Sen\,u\right) = Cos\,u\,\frac{du}{dx}}$$

Ejemplos:

1. $\frac{d}{dx}\left(Sen\,2x\right) = Cos\,2x\,\frac{d}{dx}\left(2x\right) = 2Cos\,2x$

2. $\frac{d}{dx}\left(x^2 Sen\left(x\right)\right) = x^2\frac{d}{dx}\left(Sen\left(x\right)\right) + Sen\left(x\right)\frac{d}{dx}\left(x^2\right) = x^2 Cos\left(x\right) + \left(2x\right)Sen\left(x\right)$

3. $\frac{d}{dx}\left(\frac{Sen\left(x\right)}{x}\right) = \frac{x\frac{d}{dx}\left(Sen\left(x\right)\right) - Sen\left(x\right)\frac{d}{dx}\left(x\right)}{x^2} = \frac{xCos\left(x\right) - Sen\left(x\right)}{x^2}$

4. $\frac{d}{dx}\left(\frac{Sen^2\left(x\right)}{x^2}\right) = \frac{x^2\frac{d}{dx}\left(Sen^2\left(x\right)\right) - Sen^2\left(x\right)\frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2} =$

$$= \frac{x^2\left(2Sen\left(x\right)Cos\left(x\right)\right) - \left(2x\right)Sen^2\left(x\right)}{x^4} =$$

$$= \frac{2x^2 Sen\left(x\right)Cos\left(x\right) - 2xSen^2\left(x\right)}{x^4} =$$

$$= \frac{2\not{x}\left(xSen\left(x\right)Cos\left(x\right) - Sen^2\left(x\right)\right)}{x^{\not{4}}} = \frac{2\left(xSen\left(x\right)Cos\left(x\right) - Sen^2\left(x\right)\right)}{x^3}$$

Derivada de $y = Cos\left(u\right)$

Para obtener esta derivada hay que tener presente las siguientes identidades:

$$Cos\ u = sen\left(\frac{\pi}{2} - u\right) \qquad Sen\ u = Cos\left(\frac{\pi}{2} - u\right)$$

Luego:

$$\frac{d}{dx}Cos\ u = \frac{d}{dx}sen\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right) =$$

$$= Sen\ u\left(\frac{-du}{dx}\right) = -Sen\ u\frac{du}{dx}$$

De donde se puede concluir:

$$\boxed{\frac{d}{dx}(Cos\ u) = -Sen\ u\frac{du}{dx}}$$

Ejemplos:

1. $\frac{d}{dx}(Cos\ 3x) = -Sen\ 3x\frac{d}{dx}(3x) = -3Sen\ 3x$
2. $\frac{d}{dx}\left(2x^4 + 3Cos(x)\right) = \frac{d}{dx}\left(2x^4\right) + \frac{d}{dx}\left(3Cos(x)\right) = 8x^3 - 3Sen(x)$
3. $\frac{d}{dx}\left(x^2Cos(x)\right) = x^2\frac{d}{dx}\left(Cos(x)\right) + Cos(x)\frac{d}{dx}\left(x^2\right) = -x^2Sen(x) + 2xCos(x)$
4. $$\frac{d}{dx}\left(\frac{Sen(x)}{1 - Cos(x)}\right) = \frac{\left(1 - Cos(x)\right)\frac{d}{dx}\left(Sen(x)\right) - \left(Sen(x)\right)\frac{d}{dx}\left(1 - Cos(x)\right)}{\left(1 - Cos(x)\right)^2} =$$

$$= \frac{\left(1 - Cos(x)\right)\left(Cos(x)\right) - \left(Sen(x)\right)\left(Sen(x)\right)}{\left(1 - Cos(x)\right)^2} =$$

$$= \frac{Cos(x) - Cos^2(x) - Sen^2(x)}{\left(1 - Cos(x)\right)^2} = \frac{\left(Cos(x) - 1\right)}{\left(1 - Cos(x)\right)^2} =$$

$$= \frac{(-1)\ \cancel{\left(1 - Cos(x)\right)}}{\cancel{\left(1 - Cos(x)\right)}\left(1 - Cos(x)\right)} = \frac{1}{\left(Cos(x) - 1\right)}$$

Derivada de $y = Tan(x)$

$$\frac{d}{dx}(Tan\ x)=\frac{d}{dx}\left(\frac{Sen\ x}{Cos\ x}\right)$$ *Definición función tangente*

$$\frac{d}{dx}(Tan\ x)=\frac{Cos\ x\frac{d}{dx}(Sen\ x)-Sen\ x\frac{d}{dx}(Cos\ x)}{Cos^2 x}$$ *Derivada de un cociente*

$$\frac{d}{dx}(Tan\ x)=\frac{Cos\ x\ Cos\ x-Sen\ x\ Sen\ x}{Cos^2 x}$$ *Resolviendo la derivada*

$$\frac{d}{dx}(Tan\ x)=\frac{Cos^2 x-Sen^2 x}{Cos^2 x}=\frac{1}{Cos^2 x}=Sec^2 x$$ *Agrupando términos*

De manera que si ***u*** es una función diferenciable de ***x***, aplicando la regla de la cadena a la función $y=\tan u$ se puede concluir:

$$\frac{d}{dx}Tan\ u=Sec^2 u\frac{du}{dx}$$

Ejemplos :

1. $\frac{d}{dx}Tan(5x)=Sec^2(5x)\left(\frac{d}{dx}5x\right)=5Sec^2(5x)$

2. $\frac{d}{dx}\left(x^2Tan(x)\right)=x^2\frac{d}{dx}\left(Tan(x)\right)+Tan(x)\frac{d}{dx}\left(x^2\right)=$
$x^2Sec^2(x)+Tan(x)(2x)=x\left(xSec^2(x)+2Tan(x)\right)$

3. $\frac{d}{dx}\left(-x+Tan(x)\right)=\frac{d}{dx}(-x)+\frac{d}{dx}\left(Tan(x)\right)=-1+Sec^2(x)$

4. $\frac{d}{dx}\left(x^2Tan\left(1/x\right)\right)=x^2\frac{d}{dx}\left(Tan\left(1/x\right)\right)+Tan\left(1/x\right)\frac{d}{dx}\left(x^2\right)=$
$x^2\left[Sec^2\left(1/x\right)\frac{d}{dx}\left(1/x\right)\right]+\left[Tan\left(1/x\right)\right](2x)=$
$\left(\cancel{x^2}\right)\left(\frac{1}{\cancel{x^2}}\right)Sec^2\left(1/x\right)+(2x)Tan\left(1/x\right)=$
$Sec^2\left(1/x\right)+(2x)Tan\left(1/x\right)$

5. $$\frac{d}{dx}\left(Tan\left(\pi x+1\right)\right)=Sec^2\left(\pi x+1\right)\frac{d}{dx}\left(\pi x+1\right)=$$
$$\left[Sec^2\left(\pi x+1\right)\right]\left(\pi\right)=\pi\left(Sec^2\left(\pi x+1\right)\right)$$

Derivada de $y=Cot\left(u\right)$

$$\frac{d}{dx}Cot\ x=\frac{d}{dx}\left(\frac{Cos\ x}{Sen\ x}\right)$$ *Definición de Cotangente*

$$\frac{d}{dx}Cot\ x=\frac{Sen\ x\frac{d}{dx}\left(Cos\ x\right)-Cos\ x\frac{d}{dx}\left(Sen\ x\right)}{Sen^2 x}$$ *Derivada de un cociente*

$$\frac{d}{dx}Cot\ x=\frac{-Sen\ x\ Sen\ x-Cos\ x\ Cos\ x}{Sen^2 x}$$ *Resolviendo la derivada*

$$\frac{d}{dx}\left(Cot\ x\right)=\frac{-Sen^2 x-Cos^2 x}{Sen^2 x}=\frac{-1}{Sen^2 x}=-Csc^2 x$$ *Factorizando y Simplifando*

De manera que si ***u*** es una función diferenciable de ***x***, aplicando la regla de la cadena a la función $y=Cot\ u$ se puede concluir:

$$\frac{d}{dx}Cot\ u=-Csc^2 u\frac{du}{dx}$$

Ejemplos :

1. $$\frac{d}{dx}\left(Cot^2\left(x\right)\right)=2\left(Cot\left(x\right)\right)\frac{d}{dx}\left(Cot\left(x\right)\right)=2\left(Cot\left(x\right)\right)\left(-Csc^2\left(x\right)\right)=$$
$$-2Cot\left(x\right)Csc^2\left(x\right)$$

2. $$\frac{d}{dx}\left(Cot^2\left(5x\right)\right)=2\left(Cot\left(5x\right)\right)\frac{d}{dx}\left(Cot\left(5x\right)\right)=2\left(Cot\left(5x\right)\right)\left(-Csc^2\left(5x\right)\right)\left(\frac{d}{dx}5x\right)=$$
$$2\left(Cot\left(5x\right)\right)\left(-Csc^2\left(5x\right)\right)\left(5\right)=-10\left[Cot\left(5x\right)\right]\left[Csc^2\left(5x\right)\right]$$

3. $$\frac{d}{dx}\left(\frac{1}{Ctg\left(3x^2\right)}\right)=\frac{Ctg\left(3x^2\right)\left(\cancelto{0}{\frac{d}{dx}(1)}\right)-(1)\frac{d}{dx}\left(Ctg\left(3x^2\right)\right)}{\left(Ctg\left(3x^2\right)\right)^2}=$$

$$\frac{-\left(-Csc^2\left(3x^2\right)\right)\frac{d}{dx}\left(3x^2\right)}{Ctg^2\left(3x^2\right)}=\frac{\left(Csc^2\left(3x^2\right)\right)(6x)}{Ctg^2\left(3x^2\right)}=\frac{6x\left(Csc^2\left(3x^2\right)\right)}{Ctg^2\left(3x^2\right)}$$

4. $$\frac{d}{dx}\left(-x+Ctg(x)\right)=\frac{d}{dx}(-x)+\frac{d}{dx}\left(Ctg(x)\right)=-1+\left(-Csc^2(x)\right)=-1-Csc^2(x)$$

5. $$\frac{d}{dx}\left(x^2Ctg\left(\frac{1}{x}\right)\right)=x^2\frac{d}{dx}\left(Ctg\left(\frac{1}{x}\right)\right)+Ctg\left(\frac{1}{x}\right)\frac{d}{dx}\left(x^2\right)=$$

$$x^2\left(Csc^2\left(\frac{1}{x}\right)\right)\frac{d}{dx}\left(\frac{1}{x}\right)+Ctg\left(\frac{1}{x}\right)\frac{d}{dx}\left(x^2\right)=$$

$$\cancel{x^2}\left(Csc^2\left(\frac{1}{x}\right)\right)\left(\frac{-1}{\cancel{x^2}}\right)+Ctg\left(\frac{1}{x}\right)(2x)=$$

$$Csc^2\left(\frac{1}{x}\right)+(2x)Ctg\left(\frac{1}{x}\right)$$

Derivada de $y=Sec(u)$

$$\frac{d}{dx}\left(Sec(x)\right)=\frac{d}{dx}\left(\frac{1}{Cos(x)}\right)$$ *Definición de Secante*

$$\frac{d}{dx}\left(Sec(x)\right)=\frac{Cos(x)\frac{d}{dx}(1)-(1)\frac{d}{dx}\left(Cos(x)\right)}{Cos^2(x)}$$ *Derivada de un cociente*

$$\frac{d}{dx}\left(Sec(x)\right)=\frac{-1\left(-Sen(x)\right)}{Cos^2(x)}$$ *Resolviendo la derivada*

$$\frac{d}{dx}\left(Sec(x)\right)=\frac{Sen(x)}{Cos^2(x)}=\frac{Sen(x)}{Cos(x)}\frac{-1}{Cos(x)}=Tan(x)\,Sec(x)$$ *Simplificando y factorizado*

De manera que si ***u*** es una función diferenciable de ***x***, aplicando la regla de la cadena a la función $y=Sec\,u$ se puede concluir:

$$\boxed{\frac{d}{dx}Sec(u)=Tan(u)\,Sec(u)\frac{du}{dx}}$$

Ejemplos :

1. $\frac{d}{dx}\left(Sec^2(x)\right)=2\left(Sec(x)\right)\frac{d}{dx}\left(Sec(x)\right)=2\left(Sec(x)\right)\left(Sec(x)Tan(x)\right)=$

$2Sec^2(x)Tan(x)$

2. $\frac{d}{dx}\left(Sec^2(5x)\right)=2\left(Sec(5x)\right)\frac{d}{dx}\left(Sec(5x)\right)=$

$2\left(Sec(5x)\right)\left(Sec(5x)Tan(5x)\right)\left(\frac{d}{dx}5x\right)=$

$2\left(Sec(5x)\right)\left(Sec(5x)Tan(5x)\right)(5)=$

$10\left(Sec^2(5x)\right)\left(Tan(5x)\right)$

3. $\frac{d}{dx}\left(\frac{1}{Sec\left(x^2\right)}\right)=\frac{Sec\left(x^2\right)\left(\frac{d}{dx}(1)\right)^{0}-(1)\frac{d}{dx}\left(Sec\left(x^2\right)\right)}{\left(Sec\left(x^2\right)\right)^2}=$

$\frac{-\left(Sec\left(x^2\right)Tan\left(x^2\right)\right)\frac{d}{dx}\left(x^2\right)}{\left(Sec\left(x^2\right)\right)^2}=$

$\frac{-\left(Sec\left(x^2\right)Tan\left(x^2\right)\right)(2x)}{\left(Sec\left(x^2\right)\right)^2}=$

$\frac{-(2x)\left(Sec\left(x^2\right)Tan\left(x^2\right)\right)}{\left(Sec\left(x^2\right)\right)^2}=$

$\frac{-(2x)Tan\left(x^2\right)}{Sec\left(x^2\right)}=$

4. $\frac{d}{dx}\left(x^2+Sec(x)\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(Sec(x)\right)=2x+\left(Sec(x)Tan(x)\right)$

5. $\frac{d}{dx}\left(x^2Sec(3x)\right)=x^2\frac{d}{dx}\left(Sec(3x)\right)+Sec(3x)\frac{d}{dx}\left(x^2\right)=$

$$x^2\left(Sec(3x)Tan(3x)\right)\frac{d}{dx}(3x)+Sec(3x)(2x)=$$
$$x^2\left(Sec(3x)Tan(3x)\right)(3)+(2x)Sec(3x)=$$
$$3x^2\left(Sec(3x)Tan(3x)\right)+(2x)Sec(3x)$$

Derivada de $y=Csc(u)$

$$\frac{d}{dx}Csc(x)=\frac{d}{dx}\left(\frac{1}{Sen(x)}\right)$$ *Definición de la Cosecante*

$$\frac{d}{dx}Csc(x)=\frac{Sen(x)\frac{d}{dx}(1)-1\frac{d}{dx}\left(Sen(x)\right)}{Sen^2(x)}$$ *Derivada de un Cociente*

$$\frac{d}{dx}Csc(x)=\frac{-1\left(Cos(x)\right)}{Sen^2(x)}$$ *Resolviendo la derivada.*

$$\frac{d}{dx}Csc(x)=\frac{-Cos(x)}{Sen^2(x)}=\frac{-Cos(x)}{Sen(x)}\frac{1}{Sen(x)}=$$

$$\frac{d}{dx}Csc(x)=-Cot(x)Csc(x)$$ *Simplificando y factorizando*

De manera que si ***u*** es una función diferenciable de ***x***, aplicando la regla de la cadena a la función $y=Csc(u)$ se puede concluir:

$$\frac{d}{dx}Csc(u)=-Cot(u)Csc(u)\frac{du}{dx}$$

Ejemplos :

1. $$\frac{d}{dx}\left(Csc^2(x)\right)=2\left(Csc(x)\right)\frac{d}{dx}\left(Csc(x)\right)=2\left(Csc(x)\right)\left(-Csc(x)Ctg(x)\right)=$$
$$2\left(Csc(x)\right)\left(-Csc(x)Ctg(x)\right)=-2\left(Csc^2(x)\right)\left(Ctg(x)\right)$$

2. $$\frac{d}{dx}\left(Csc^2\left(x^3\right)\right)=2\left(Csc\left(x^3\right)\right)\frac{d}{dx}\left(Csc\left(x^3\right)\right)=$$

$$2\left(Csc\left(x^3\right)\right)\left(-Csc\left(x^3\right)Ctg\left(x^3\right)\right)\frac{d}{dx}\left(x^3\right)=$$

$$-2\left(Csc\left(x^3\right)\right)\left(Csc\left(x^3\right)Ctg\left(x^3\right)\right)\left(3x^2\right)=$$

$$-6x^2\left(Csc^2\left(x^3\right)\right)\left(Ctg\left(x^3\right)\right)=-6x^2\frac{1}{Sen^2\left(x^3\right)}\frac{Cos\left(x^3\right)}{Sen\left(x^3\right)}$$

$$-6x^2\frac{Cos\left(x^3\right)}{Sen^3\left(x^3\right)}$$

3. $$\frac{d}{dx}\left(\frac{1}{Csc(x)}\right)=\frac{Csc(x)\left(\cancelto{0}{\frac{d}{dx}(1)}\right)-(1)\frac{d}{dx}\left(Csc(x)\right)}{\left(Csc(x)\right)^2}=$$

$$\frac{-(1)\left(Csc(x)Ctg(x)\right)}{\left(Csc(x)\right)^2}=\frac{-(1)\left(\cancel{Csc(x)}Ctg(x)\right)}{\left(Csc(x)\right)^{\cancel{2}}}$$

$$\frac{-\left(Ctg(x)\right)}{\left(Csc(x)\right)}=\frac{-Cos(x)}{\cancel{Sen(x)}}\frac{\cancel{Sen(x)}}{1}=-Cos(x)$$

4. $$\frac{d}{dx}\left(x+Csc(x)\right)=\frac{d}{dx}(x)+\frac{d}{dx}\left(Csc(x)\right)=x+\left(Csc(x)Ctg(x)\right)$$

5. $$\frac{d}{dx}\left(x^2Csc(3x)\right)=x^2\frac{d}{dx}\left(Csc(3x)\right)+Csc(3x)\frac{d}{dx}\left(x^2\right)=$$

$$x^2\left(-Csc(3x)Ctg(3x)\right)\frac{d}{dx}(3x)+Csc(3x)(2x)=$$

$$-x^2\left(Csc(3x)Ctg(3x)\right)(3)+Csc(3x)(2x)=$$

$$-\left(3x^2\right)\left(Csc(3x)Ctg(3x)\right)+(2x)Csc(3x)=$$

Ejercicios Propuestos :

Encontrar la derivada de las siguientes expresiones:

1. $y=x-3\,Sen(x)$

2. $y=Cos(x)-2\,Tan(x)$

3. $y=t^3Cos(t)$

4. $y=4\,Sec(t)+Tan(t)$

5. $y = \dfrac{Tan(x)}{x}$

6. $y = \dfrac{sen(x)}{1 + Cos(x)}$

7. $y = \dfrac{x}{Sen(x) + Cos(x)}$

8. $y = \dfrac{Tan(x) - 1}{Sec(x)}$

9. $y = \dfrac{Sen(x)}{x^2}$

10. $y = \dfrac{Sen^2(x)}{x}$

11. $y = x(Cos(x))(Sen(x))$

12. $y = Sen^3(2x)$

13. $y = Sen^2(x) + Cos^2(x)$

14. $y = \dfrac{x^2 + 1}{xSen(x)}$

15. $y = sen^4(x^2 + 3x)$

16. $y = xSen^2(2x)$

17. $y = Sen^3(Cos(t))$

18. $y = Sen(Cos^2(x))$

19. $y = x^2 Sen^5(2x)$

20. $y = Sec(x)Sen(x)$