On n-dimensional Hilbert transform of weighted distributions

MARTHA GUZMÁN-PARTIDA*
Departamento de Matemáticas, Universidad de Sonora,
Hermosillo, Sonora 83000, México

Abstract

We define a family of conjugate Poisson transforms of distributions T in the optimal space of weighted distributions $w_1...w_n\mathcal{D}'_{L^1}$, by means of the $\mathcal{S}'$-convolution. We prove that their boundary values in the topology of this space of distributions are of the form $\mathcal{H}(T)$, where $\mathcal{H}$ is the n-dimensional Hilbert transform of T.

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1 Introduction and notation

The n-dimensional Hilbert transform $\mathcal{H}$ is a convolution operator with the kernel $p.v.\frac{1}{x_1} \otimes ... \otimes p.v.\frac{1}{x_n}$ that is defined for a certain class of tempered distributions. This class can be easily identified. In fact, the optimal space of tempered distributions for which the n-dimensional Hilbert transform is defined is the class of weighted integrable distributions $w_1...w_n\mathcal{D}'_{L^1}$ (see [10],[1]). The convolution considered here, is the so-called $\mathcal{S}'$-convolution, a commutative operation for tempered distributions developed by Y. Hirata and H. Ogata [8] and R. Shiraishi [12] with the purpose of extending the validity of the Fourier exchange formula $\mathcal{F}(S * T) = \mathcal{F}(S)\,\mathcal{F}(T)$, where the product on the right-hand side must be understood in an appropriate sense that we will precise later.

The aim of this paper is to solve the following boundary value problem: given a distribution $T \in w_1...w_n\mathcal{D}'_{L^1}$, we consider a family of conjugate Poisson transforms of T and prove that its boundary value in the topology of $w_1...w_n\mathcal{D}'_{L^1}$ is $\mathcal{H}(T)$. Thus, we generalize a result proved by Pandey and Singh in ([9, Th. 4.3]) for the space of distributions $\mathcal{D}'_{L^p}$, $1 < p < \infty$.

The paper is divided in three sections. In Section 2 we consider two family of kernels: the product domain version of the one-dimensional Poisson kernel

*Corresponding author. Email: martha@gauss.mat.uson.mx

and the product domain version of the one-dimensional conjugate Poisson kernel. Both versions correspond to the domain $\mathbb{R}_+^2 \times ... \times \mathbb{R}_+^2$ consisting of n copies of the upper half-plane and they are related by means of the one-dimensional Hilbert transform. We present here some results concerning to the $\mathcal{S}'$-convolution with these kernels. In Section 3 we consider the n-dimensional Hilbert transform of distributions in $w_1 ... w_n \mathcal{D}'_{L^1}$ and we prove the main results of this work.

Finally, concerning notation, partial derivatives will be denoted as ∂^α, where α is a multi-index $(\alpha_1, ..., \alpha_n)$. We shall use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$. For a function g, we will indicate with $\overset{\vee}{g}$ the function $x \to g(-x)$. Given a distribution T, we will denote with $\check{T}$ the distribution $\varphi \to \left(T, \overset{\vee}{\varphi}\right)$, where φ is an appropriate test function. The Fourier transform will be denoted as $\mathcal{F}$. The letter C will indicate a positive constant, probably different at different occurrences.

2 Poisson and conjugate Poisson transform of weighted distributions

In this section we will consider the Poisson and conjugate Poisson transforms of certain weighted integrable distributions. These transforms are defined by means of the $\mathcal{S}'$-convolution with the product domain versions of the one-dimensional Poisson and conjugate Poisson kernels, respectively.

Let us first review the notion of $\mathcal{S}'$-convolution that will be used. The spaces of functions and distributions related to this notion can be found in [11] and [6]. For the sake of completeness we give a brief account of them.

The space of integrable distributions $\mathcal{D}'_{L^1}$ is, by definition, the strong dual of the space $\dot{B}$ of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that $\partial^\alpha \varphi \to 0$ as $|x| \to \infty$, for each multi-index α. $\dot{B}$ is a closed subspace of the space B consisting of those smooth functions φ with the property that $\partial^\alpha \varphi$ is bounded for every multi-index α, endowed with the topology of the uniform convergence on $\mathbb{R}^n$ of each derivative. The space C_0^∞ of compactly supported smooth functions defined on $\mathbb{R}^n$, is dense in $\dot{B}$ but not in B. Each $T \in \mathcal{D}'_{L^1}$ can be represented as $T = \sum_{finite} \partial^\alpha f_\alpha$, where $f_\alpha \in L^1$. Consequently, we have the strict inclusions $\mathcal{E}' \subset \mathcal{D}'_{L^1} \subset \mathcal{S}'$.

It is possible to consider $\mathcal{D}'_{L^1}$ as the strong dual of the space B, provided that we endow B with a topology that gives rise to the following notion of sequence convergence: a sequence $\{\varphi_j\}$ converges to φ if, for each multi-index α, $\sup_j \|\partial^\alpha \varphi_j\|_\infty < \infty$ and the sequence $\{\partial^\alpha \varphi_j\}$ converges to $\partial^\alpha \varphi$ uniformly on compact sets. If we denote as B_c the resulting topological space, it can be proved that C_0^∞, and so $\dot{B}$, is dense in B_c. Thus, it turns out that $\mathcal{D}'_{L^1}$ is also the dual of B_c.

Other spaces that will be of interest later are the following:

For $1 \leq p < \infty$, let

$$\mathcal{D}_{L^p} = \{\varphi \in C^\infty : \partial^\alpha \varphi \in L^p \text{ for every multi-index } \alpha\},$$

endowed with the topology defined by the family of norms

$$\|\varphi\|_{m,p} = \left[\sum_{|\alpha|\leq m} \|\partial^\alpha \varphi\|_p^p\right]^{1/p}, \; m = 0, 1, 2, ...$$

The space $\mathcal{D}_{L^p}$ is a Fréchet space and we have the dense and continuous strict inclusions $C_0^\infty \subset \mathcal{D}_{L^p} \subset \mathcal{D}'$.

The space $\mathcal{D}_{L^\infty}$ is the space B defined above. For $1 \leq p < q \leq \infty$ we have the continuous strict inclusions $\mathcal{D}_{L^p} \subset \mathcal{D}_{L^q}$.

When $1 < p \leq \infty$, we denote with $\mathcal{D}'_{L^p}$ the topological dual of $\mathcal{D}_{L^{p'}}$, endowed with the strong dual topology. We have the continuous strict inclusions $\mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q}$ for $p < q$. Moreover, every distribution $T \in \mathcal{D}'_{L^p}$ can be represented as $T = \sum_{finite} \partial^\alpha f_\alpha$, $f_\alpha \in L^p$. Thus $\mathcal{D}'_{L^p}$ is continuously included in $\mathcal{S}'$.

Now, we define the notion of $\mathcal{S}'$-convolution.

Definition 1 *([8],[12]) Given two tempered distributions T and S, it is said that the $\mathcal{S}'$-convolution of T and S exists if $T\left(\check{S} * \varphi\right) \in \mathcal{D}'_{L^1}$ for every $\varphi \in \mathcal{S}$. When the $\mathcal{S}'$-convolution exists, the map*

$$\begin{aligned} &\mathcal{S} \to \mathbb{C} \\ &\varphi \to \left(T\left(\check{S} * \varphi\right), 1\right)_{\mathcal{D}'_{L^1}, B_c} \end{aligned}$$

defines a tempered distribution which will be denoted by $T * S$.

This is a commutative operation. Moreover, Definition 1 coincides with the classical definition in all the cases in which the latter makes sense.

Remark 2 *Y. Hirata and H. Ogata in [8] introduced the $\mathcal{S}'$-convolution to extend the validity of the Fourier exchange formula (originally proved by L. Schwartz in [11] for pairs of distributions in $\mathcal{O}'_c \times \mathcal{S}'$): given T, $S \in \mathcal{S}'$ such that the $\mathcal{S}'$-convolution $T * S$ is defined, then*

$$\mathcal{F}\left(T * S\right) = \mathcal{F}(T)\mathcal{F}(S). \tag{1}$$

The product on the right hand side of (1) is understood in the following sense: for any two δ-sequences $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$, the sequences $\{(\mathcal{F}\left(T\right) * \varphi_k)\,\mathcal{F}\left(S\right)\}_{k=1}^\infty$ and $\{\mathcal{F}\left(T\right)(\mathcal{F}\left(S\right) * \psi_k)\}_{k=1}^\infty$ converge in $\mathcal{D}'$ to the same distribution and this common limit is denoted by $\mathcal{F}\left(T\right)\mathcal{F}\left(S\right)$.

As it is known, a δ-sequence is a sequence $\{\varphi_k\}_{k=1}^\infty$ of non-negative functions in C_0^∞ with the following properties:

1. *Supp φ_k converges to 0 when $k \to \infty$.*
2. *$\int \varphi_k = 1$ for every k.*

We will consider the following two n-dimensional kernels, namely:

The product domain version of the one-dimensional Poisson kernel

$$\mathcal{P}_{(y)}(x) = \prod_{j=1}^{n} P_{y_j}(x_j), \tag{2}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n)$, $(y) > (0)$, meaning that $y_1 > 0, ..., y_n > 0$, and $P_{y_j}(x_j) = \frac{1}{y_j} P_1(x_j/y_j)$, with $P_1(x_j) = \frac{1}{\pi}\frac{1}{1+x_j^2}$, $j = 1, ..., n$. This product domain version of the Poisson kernel corresponds to the domain $\mathbb{R}_+^2 \times ... \times \mathbb{R}_+^2$ consisting of n copies of the upper half-plane.

The product domain version of the one-dimensional conjugate Poisson kernel

$$\mathcal{Q}_{(y)}(x) = \prod_{j=1}^{n} Q_{y_j}(x_j) \tag{3}$$

for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $(y) > (0)$ and $Q_{y_j}(x_j) = \frac{1}{y_j} Q_1(x_j/y_j)$, with $Q_1(x_j) = \frac{1}{\pi}\frac{x_j}{1+x_j^2}$, $j = 1, ..., n$.

It is well known the fact that for $y_j > 0$, $j = 1, ..., n$

$$Q_{y_j}(x_j) = H\left(P_{y_j}\right)(x_j) \tag{4}$$

where $H\left(P_{y_j}\right)$ is the classical one-dimensional Hilbert transform of P_{y_j}, that is, $H\left(P_{y_j}\right) = \left(\frac{1}{\pi} p.v. \frac{1}{x_j}\right) * P_{y_j}$.

The optimal spaces for $\mathcal{S}'$-convolution with the kernels (2) and (3) are described in the following lines.

Definition 3 *([10],[1],[3]) Let* $w_j = \left(1 + x_j^2\right)^{1/2}$, $j = 1, ..., n$. *Then*

$$w_1...w_n \mathcal{D}'_{L^1} = \left\{T \in \mathcal{D}' : w_1^{-1}...w_n^{-1} T \in \mathcal{D}'_{L^1}\right\}$$

with the topology induced by the map

$$\begin{array}{ccc} w_1...w_n \mathcal{D}'_{L^1} & \longrightarrow & \mathcal{D}'_{L^1} \\ T & \longrightarrow & w_1^{-1}...w_n^{-1} T. \end{array}$$

Similarly, is defined the space $w_1^2...w_n^2 \mathcal{D}'_{L^1}$.

It can be proved the following characterization for these spaces.

Lemma 4 *([4], [5])*

$$w_1...w_n \mathcal{D}'_{L^1} = \left\{T \in \mathcal{S}' : T = \sum_{finite} \partial^\alpha f_\alpha,\ f_\alpha \in L^1\left(w_1^{-1}...w_n^{-1}\right)\right\}$$

$$w_1^2...w_n^2 \mathcal{D}'_{L^1} = \left\{T \in \mathcal{S}' : T = \sum_{finite} \partial^\alpha f_\alpha,\ f_\alpha \in L^1\left(w_1^{-2}...w_n^{-2}\right)\right\}.$$

Consequently, both spaces of distributions are closed under differentiation. They also contain known spaces of distributions, namely:

Proposition 5 *For $1 \leq p < \infty$, $\mathcal{D}'_{L^p} \subset w_1...w_n\mathcal{D}'_{L^1} \subset w_1^2...w_n^2\mathcal{D}'_{L^1}$ and the inclusions are strict and continuous.*

Proof. According to Lemma 4 and the characterization for the space $\mathcal{D}'_{L^p}$, the inclusions are immediate since $L^p \hookrightarrow L^1\left(w_1^{-1}...w_n^{-1}\right) \hookrightarrow L^1\left(w_1^{-2}...w_n^{-2}\right)$. That the inclusions are strict, is shown by the following examples.

Let T be the tempered distribution defined by the function

$$f(x) = \left(1+x_1^2\right)^{\beta/2}...\left(1+x_n^2\right)^{\beta/2}$$

where $0 < \beta < 1$. Since $w_1^{-2}...w_n^{-2}T \in L^1(\mathbb{R}^n)$, this shows that $T \in w_1^2...w_n^2\mathcal{D}'_{L^1}$. However, $T \notin w_1...w_n\mathcal{D}'_{L^1}$ because if we consider the sequence $\eta_j(x) = \eta(x_1/j)...\eta(x_n/j)$, $j = 1,...,n$, where $\eta \in C_0^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $|t| < 1$ and $\eta(t) = 0$ for $|t| > 2$, we can easily see that $\eta_j \to 1$ in B_c but

$$\begin{aligned}\left(w_1^{-1}...w_n^{-1}T, \eta_j\right)_{\mathcal{S}',\mathcal{S}} &= \int_{-\infty}^{\infty} \frac{\eta(x_1/j)}{\left(1+x_1^2\right)^{(1-\beta)/2}}dx_1...\int_{-\infty}^{\infty} \frac{\eta(x_n/j)}{\left(1+x_n^2\right)^{(1-\beta)/2}}dx_n \\ &\geq \int_0^j \frac{1}{\left(1+x_1^2\right)^{(1-\beta)/2}}dx_1...\int_0^j \frac{1}{\left(1+x_n^2\right)^{(1-\beta)/2}}dx_n \\ &\longrightarrow \infty \text{ as } j \to \infty\end{aligned}$$

since $0 < 1-\beta < 1$. This shows that the second inclusion is strict.

To see the first inclusion is proper, let us consider first the case $p = 1$. We can take the distribution $T = w_1^{-1}...w_n^{-1}$ which clearly belongs to $w_1...w_n\mathcal{D}'_{L^1}$ since $w_1^{-1}...w_n^{-1}T \in L^1(\mathbb{R}^n)$, however, taking a sequence $(\eta_j)_{j=1}^\infty$ as above, we have that $\eta_j \to 1$ in B_c but

$$\begin{aligned}\langle T, \eta_j\rangle &\geq \int_0^j \frac{1}{\left(1+x_1^2\right)^{1/2}}dx_1...\int_0^j \frac{1}{\left(1+x_n^2\right)^{1/2}}dx_n \\ &\longrightarrow \infty \text{ as } j \to \infty.\end{aligned}$$

Now, consider the case $1 < p < \infty$. Let q and β be such that $1/p + 1/q = 1$, $0 < \beta < 1$, and pick θ such that $1/q < \theta < 1-\beta$. Let T be the distribution $T = \left(1+x_1^2\right)^{-\beta/2}...\left(1+x_n^2\right)^{-\beta/2}$. Since $w_1^{-1}...w_n^{-1}T = \left(1+x_1^2\right)^{-(1+\beta)/2}...\left(1+x_n^2\right)^{-(1+\beta)/2} \in L^1(\mathbb{R}^n)$ is clear that $T \in w_1...w_n\mathcal{D}'_{L^1}$. For $\eta \in C_0^\infty(\mathbb{R})$ as above, let us consider the sequence $(\phi_j)_{j=1}^\infty$ defined by

$$\phi_j(x) = \frac{1}{j^\theta}\eta(x_1/j)...\frac{1}{j^\theta}\eta(x_n/j).$$

The sequence $(\phi_j)_{j=1}^{\infty}$ converges to 0 in $\mathcal{D}_{L^q}$ since for every multi-index α

$$\begin{aligned}\|\partial^{\alpha}\phi_j\|_q^q &= \frac{j^n}{j^{(\theta n+|\alpha|)q}}\int_{-\infty}^{\infty}|\partial^{\alpha_1}\eta(t_1)|^q\,dt_1...\int_{-\infty}^{\infty}|\partial^{\alpha_n}\eta(t_n)|^q\,dt_n\\ &= \frac{1}{j^{n(\theta q-1)+|\alpha|q}}\|\partial^{\alpha_1}\eta\|_q^q...\|\partial^{\alpha_n}\eta\|_q^q\\ &\longrightarrow 0 \text{ as } j\to\infty\end{aligned}$$

because $\theta q>1$. Thus $\|\phi_j\|_{m,q}\to 0$ as $j\to\infty$. However, for $j>1$ we have

$$\begin{aligned}\langle T,\eta_j\rangle &\geq \frac{1}{j^{\theta}}\int_1^j\frac{1}{(1+x_1^2)^{\beta/2}}dx_1...\frac{1}{j^{\theta}}\int_1^j\frac{1}{(1+x_n^2)^{\beta/2}}dx_n\\ &\geq C_{\beta}\left[\frac{1}{j^{\theta}}\int_1^j\frac{dx_1}{x_1^{\beta}}...\frac{1}{j^{\theta}}\int_1^j\frac{dx_n}{x_n^{\beta}}\right]\\ &= C_{\beta}\left[j^{1-\beta-\theta}-\frac{1}{j^{\theta}}\right]^n\\ &\longrightarrow \infty \text{ as } j\to\infty\end{aligned}$$

since $1-\beta-\theta>0$.

This concludes the proof of the Proposition. ■

The authors of [3] have determined the optimal spaces of tempered distributions $\mathcal{S}'$-convolvable with the kernel $\mathcal{P}_{(y)}$. In fact, they prove:

Theorem 6 *([3]) Given $T\in\mathcal{S}'$, the following statements are equivalent:*

i) $T\in w_1^2...w_n^2\mathcal{D}'_{L^1}$.

ii) *T is $\mathcal{S}'$-convolvable with $\mathcal{P}_{(y)}$ for each $(y)>(0)$.*

Also, L. Schwartz in [10] and Alvarez and Carton-Lebrun in [1] characterized the optimal space of tempered distributions $\mathcal{S}'$-convolvable with the n-dimensional Hilbert kernel $p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}$, namely,

Theorem 7 *([1]) Let $T\in\mathcal{S}'$. Then, the following statements are equivalent:*

a) $T\in w_1...w_n\mathcal{D}'_{L^1}$.

b) *T is $\mathcal{S}'$-convolvable with $p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}$.*

In view of (4) and Theorem 7, it seems that the optimal space for $\mathcal{S}'$-convolution with the kernel $\mathcal{Q}_{(y)}$ is $w_1...w_n\mathcal{D}'_{L^1}$.This can be easily shown just appealing to the following generalization of Theorem 7 proved in [2, Th. 29].

Theorem 8 *([2]) Let $T\in\mathcal{S}'$. Then, the following statements are equivalent:*

a) $T\in w_1...w_n\mathcal{D}'_{L^1}$.

b) *T is $\mathcal{S}'$-convolvable with the class $\mathcal{P}$.*

The class $\mathcal{P}$ above denotes the family of distributions $p.v.k\,(x_1,...,x_n)$ where k is a kernel in $L^1_{loc}\,(\mathbb{R}^n \setminus \{x_1 = 0 \text{ or } ... \text{ or } x_n = 0\})$ with the following cancellation and size properties:

$$\int_{a<|x_j|<b} k\,(x_1,...,x_j,...,x_n)\,dx_j = 0 \text{ for each } 0 < a < b,\ j = 1,...,n \tag{5}$$

and,

$$|k\,(x_1,...,x_n)| \leq \frac{C}{|x_1|\,...\,|x_n|} \text{ for } x_j \neq 0,\ j = 1,...,n, \tag{6}$$

for some positive constant C.

The family of kernels $\mathcal{Q}_{(y)}$ with $(y) > 0$ is included in the class $\mathcal{P}$. Thus, we can state:

Theorem 9 *Let $T \in \mathcal{S}'$. Then, the following statements are equivalent:*

a) *$T \in w_1...w_n\mathcal{D}'_{L^1}$.*

b) *T is $\mathcal{S}'$-convolvable with $\mathcal{Q}_{(y)}$ for each $(y) > (0)$.*

The $\mathcal{S}'$-convolution of a distribution $T \in w_1...w_n\mathcal{D}'_{L^1}$ with the kernel $\mathcal{Q}_{(y)}$ is a function whose value at x is given by

$$T * \mathcal{Q}_{(y)}\,(x) = \left(\prod_{j=1}^{n} w_j^{-1}(t_j)T_t, \prod_{j=1}^{n} w_j(t_j)Q_{y_j}(x_j - t_j)\right)_{\mathcal{D}'_{L^1},B_c} \tag{7}$$

for each $(y) > 0$. Indeed, writing

$$\prod_{j=1}^{n} w_j^{-1}T = \sum_{finite} \partial^\alpha f_\alpha$$

with $f_\alpha \in L^1\,(\mathbb{R}^n)$, for $\varphi \in \mathcal{S}$ we have

$$\left(T * \mathcal{Q}_{(y)}, \varphi\right)_{\mathcal{S}',\mathcal{S}} = \left(w_1^{-1}...w_n^{-1}Tw_1...w_n\left(\check{\mathcal{Q}}_{(y)} * \varphi\right), 1\right)_{\mathcal{D}'_{L^1},B_c}. \tag{8}$$

The function $w_1...w_n\left(\check{\mathcal{Q}}_{(y)} * \varphi\right)$ belongs to B since $\check{\mathcal{Q}}_{(y)} * \varphi$ is a C^∞ function such that

$$\begin{aligned}
\left|\check{\mathcal{Q}}_{(y)}*\varphi(x)\right| &\leq \int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{1}{|y_j|}\frac{|x_j-t_j|/|y_j|}{1+((x_j-t_j)/y_j)^2}|\varphi(t)|\,dt \\
&\leq \int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{1}{|y_j|}\frac{\left(1+((x_j-t_j)/y_j)^2\right)^{1/2}}{1+((x_j-t_j)/y_j)^2}|\varphi(t)|\,dt \\
&\leq \int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{1}{|y_j|}\left(1+((x_j-t_j)/y_j)^2\right)^{-1/2}|\varphi(t)|\,dt \\
&\leq C\int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{1}{|y_j|}\left(1+(x_j/y_j)^2\right)^{-1/2}\left(1+(t_j/y_j)^2\right)^{1/2}\left(1+t_j^2\right)^{-2}dt \\
&\leq C\prod_{j=1}^{n}\frac{1}{|y_j|}\left(1+(x_j/y_j)^2\right)^{-1/2}\int_{-\infty}^{\infty}\left(1+(t_j/y_j)^2\right)^{1/2}\left(1+t_j^2\right)^{-2}dt_j,
\end{aligned} \tag{9}$$

hence $w_1...w_n\left(\check{\mathcal{Q}}_{(y)}*\varphi\right)$ is bounded. Moreover

$$\partial^{\alpha}\left[w_1...w_n\left(\check{\mathcal{Q}}_{(y)}*\varphi\right)\right]=\sum_{\beta\leq\alpha}C_{\alpha,\beta}\partial^{\beta_1}w_1...\partial^{\beta_n}w_n\left(\check{\mathcal{Q}}_{(y)}*\partial^{\alpha-\beta}\varphi\right)$$

and taking into account that $\partial^{\beta_j}w_j$ is w_j or a C^{∞} bounded function, proceeding as above we can show again that $\partial^{\alpha}\left[w_1...w_n\left(\check{\mathcal{Q}}_{(y)}*\varphi\right)\right]$ is bounded. Thus, we can expand the right hand side of (8) as $\left(w_1^{-1}...w_n^{-1}T, w_1...w_n\left(\check{\mathcal{Q}}_{(y)}*\varphi\right)\right)_{\mathcal{D}'_{L^1},B_c}$ and using Fubini's theorem we can finally obtain (7).

We have seen in Lemma 4 that $w_1...w_n\mathcal{D}'_{L^1}$ is closed under differentiation. Moreover we can also obtain the following results.

Lemma 10 *If $T\in w_1...w_n\mathcal{D}'_{L^1}$ then $\partial^{\alpha}\left(T*\mathcal{Q}_{(y)}\right)=(\partial^{\alpha}T)*\mathcal{Q}_{(y)}$ for each multi-index α.*

Proof. The proof of this Lemma follows exactly the same scheme than that of [5, Lemma 9] but this time using the the fact that for every $j=1,...,n$, each $\varphi\in\mathcal{S}(\mathbb{R})$ and every $k=0,1,2,...$

$$\left|Q_{y_j}*\partial^k\varphi\right|\leq C_{y_j}w_j^{-1},$$

which can be proved as we did in estimate (9). ■

Lemma 11 *Given $T\in w_1...w_n\mathcal{D}'_{L^1}$ the $\mathcal{S}'$-convolution $T*\mathcal{Q}_{(y)}$ belongs to the space $w_1...w_n\mathcal{D}_{L^1}$ for each $(y)>0$.*

Proof. By Lemmas 4 and 10, it suffices to prove that $T*\mathcal{Q}_{(y)} \in L^1\left(w_1^{-1}...w_n^{-1}\right)$ for every $T \in w_1...w_n\mathcal{D}'_{L^1}$. Once again, following step by step the scheme of the proof of [5, Lemma 10] and noticing that for $j = 1, ..., n$ we have

$$\left|\partial^{\beta_j} w_j\left(t_j\right)\right| \leq C_{\beta_j}\left(1+t_j^2\right)^{\frac{1-\beta_j}{2}}$$

and

$$\left|\partial^{\alpha_j-\beta_j} Q_{y_j}\left(x_j-t_j\right)\right| \leq \frac{C_{\alpha_j,\beta_j}}{y^{1+\alpha_j-\beta_j}}\left(1+\frac{\left(x_j-t_j\right)^2}{y_j^2}\right)^{-\frac{\left(1+\left(\alpha_j-\beta_j\right)\right)}{2}},$$

we can finally prove our assertion. ■

Remark 12 *As a consequence of Lemma 11, for each $(y) > 0$, the $\mathcal{S}'$-convolution with the kernel $\mathcal{Q}_{(y)}$ preserves the space $L^1\left(w_1^{-1}...w_n^{-1}\right)$.*

The proofs of Lemmas 10 and 11 rely on the same arguments given to prove the corresponding results for the kernel $\mathcal{P}_{(y)}$. Moreover, for this kernel it can be proved ([5, Th. 11]):

Theorem 13 *[5] Given $T \in w_1^2...w_n^2\mathcal{D}'_{L^1}$, the $\mathcal{S}'$-convolution $T*\mathcal{P}_{(y)}$ converges to T in $w_1^2...w_n^2\mathcal{D}'_{L^1}$ as $(y) \to (0)^+$.*

It is also possible to consider the family $\mathcal{K}$ of kernels that can be written as

$$P_{y_{j_1}}...P_{y_{j_k}}Q_{y_{i_1}}...Q_{y_{i_{n-k}}}$$

where $1 \leq j_1 < ... < j_k \leq n$, $1 \leq i_1 < ... < i_{n-k} \leq n$ and $\{j_1, ..., j_k\} \cap \{i_1, ..., i_{n-k}\} = \varnothing$.

Then, for $T \in w_1...w_n\mathcal{D}'_{L^1}$, the $\mathcal{S}'$-convolution $T*\left(P_{y_{j_1}}...P_{y_{j_k}}Q_{y_{i_1}}...Q_{y_{i_{n-k}}}\right)$ is defined. In fact, it is a function whose value at x is given by

$$\left(\prod_{j=1}^n w_j^{-1}(t_j)T_t, w_{j_1}(t_{j_1})P_{y_{j_1}}(x_{j_1}-t_{j_1})...w_{i_{n-k}}(t_{i_{n-k}})Q_{y_{i_{n-k}}}(x_{i_{n-k}}-t_{i_{n-k}})\right)_{\mathcal{D}'_{L^1},B_c} \tag{10}$$

Remark 14 *Given $T \in w_1...w_n\mathcal{D}'_{L^1}$ the $\mathcal{S}'$-convolution of T and any of the kernels of the class $\mathcal{K}$ belongs to the space $w_1...w_n\mathcal{D}_{L^1}$ for each $(y) > 0$.*

Indeed, given $K \in \mathcal{K}$, all that we need to notice is that $T*K \in L^1\left(w_1^{-1}...w_n^{-1}\right)$ and this can be seen as in Lemma 11 by using (10) and the estimates

$$\left|\partial^{\beta_j} w_j\left(t_j\right)\right| \leq C_{\beta_j}\left(1+t_j^2\right)^{\frac{1-\beta_j}{2}},$$

$$\left|\partial^{\alpha_j-\beta_j} P_{y_j}\left(x_j-t_j\right)\right| \leq \frac{C_{\alpha_j,\beta_j}}{y^{1+\alpha_j-\beta_j}}\left(1+\frac{\left(x_j-t_j\right)^2}{y_j^2}\right)^{-\frac{\left(2+\left(\alpha_j-\beta_j\right)\right)}{2}},$$

and

$$\left|\partial^{\alpha_j-\beta_j}Q_{y_j}\left(x_j-t_j\right)\right|\leq\frac{C_{\alpha_j,\beta_j}}{y^{1+\alpha_j-\beta_j}}\left(1+\frac{\left(x_j-t_j\right)^2}{y_j^2}\right)^{-\frac{\left(1+\left(\alpha_j-\beta_j\right)\right)}{2}},$$

$j=1,...,n.$

3 Hilbert transform of distributions in $w_1...w_n\mathcal{D}'_{L^1}$

For $T\in w_1...w_n\mathcal{D}'_{L^1}$, let us denote by $\mathcal{H}\left(T\right)$ the n-dimensional Hilbert transform of T, that is,

$$\mathcal{H}\left(T\right)=\frac{1}{\pi^n}\left(p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}\right)*T. \tag{11}$$

In [9, Th. 4.3], Pandey and Singh prove that for $T\in\mathcal{D}'_{L^p}$ the $\mathcal{S}'$-convolution $T*\mathcal{Q}_{(y)}$ converges to $\mathcal{H}\left(T\right)$ in $\mathcal{D}'_{L^p}$ in the weak sense as $(y)\to(0)^+$.

In view of this remark and results of Section 2, it is natural to think that given $T\in w_1...w_n\mathcal{D}'_{L^1}$ the $\mathcal{S}'$-convolution $T*\mathcal{Q}_{(y)}$ converges to $\mathcal{H}\left(T\right)$ in $w_1...w_n\mathcal{D}'_{L^1}$ as $(y)\to(0)^+$. The first step to prove this assertion is the following result:

Proposition 15 *If T is a distribution in $w_1...w_n\mathcal{D}'_{L^1}$ then, for every $\delta_1>0,...,\delta_n>0$ we have that $\mathcal{H}\left(T\right)\in w_1^{\delta_1}...w_n^{\delta_n}\mathcal{D}'_{L^1}$. Moreover, $\mathcal{H}\left(w_1...w_n\mathcal{D}'_{L^1}\right)$ is not included in $\mathcal{D}'_{L^1}$.*

In particular, $\mathcal{H}\left(w_1...w_n\mathcal{D}'_{L^1}\right)\subset w_1...w_n\mathcal{D}'_{L^1}$.

Proof. We first notice that for $T\in w_1...w_n\mathcal{D}'_{L^1}$ and every $\varphi\in\mathcal{S}$ we have

$$\left\langle\mathcal{H}\left(T\right),\varphi\right\rangle_{\mathcal{S}',\mathcal{S}}=\left(-1\right)^n\left\langle T,\mathcal{H}\left(\varphi\right)\right\rangle_{w_1...w_n\mathcal{D}'_{L^1},w_1^{-1}...w_n^{-1}B_c} \tag{12}$$

$$=\left(-1\right)^n\left\langle w_1^{-1}...w_n^{-1}T,w_1...w_n\mathcal{H}\left(\varphi\right)\right\rangle_{\mathcal{D}'_{L^1},B_c} \tag{13}$$

Indeed, by definition of $\mathcal{S}'$-convolution and using the fact that

$$w_1...w_n\left(\left(p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}\right)^{\vee}*\varphi\right)\in B$$

(see [2, Th. 29]) we obtain

$$\left\langle\mathcal{H}\left(T\right),\varphi\right\rangle_{\mathcal{S}',\mathcal{S}}=\frac{1}{\pi^n}\left\langle T\left(\left(p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}\right)^{\vee}*\varphi\right),1\right\rangle_{\mathcal{D}'_{L^1},B_c} \tag{14}$$

$$=\frac{1}{\pi^n}\left\langle T,\left(p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}\right)^{\vee}*\varphi\right\rangle_{w_1...w_n\mathcal{D}'_{L^1},w_1^{-1}...w_n^{-1}B_c}$$

$$=\left(-1\right)^n\left\langle T,\mathcal{H}\left(\varphi\right)\right\rangle_{w_1...w_n\mathcal{D}'_{L^1},w_1^{-1}...w_n^{-1}B_c}$$

and also, the right hand side of (14) can be written as

$$\frac{1}{\pi^n}\left\langle w_1^{-1}...w_n^{-1}T\left(w_1...w_n\left(\left(p.v.\frac{1}{x_1}\otimes...\otimes p.v.\frac{1}{x_n}\right)^{\vee}*\varphi\right)\right),1\right\rangle_{\mathcal{D}'_{L^1},B_c}$$
$$=(-1)^n\left\langle w_1^{-1}...w_n^{-1}T,w_1...w_n\mathcal{H}(\varphi)\right\rangle_{\mathcal{D}'_{L^1},B_c}.$$

This proves our claim.

Now, assume without loss of generality that $w_1^{-1}...w_n^{-1}T=\partial^\alpha f$ with $f\in L^1(\mathbb{R}^n)$.

Consider first the case $\alpha=0$.

To prove that $\mathcal{H}(T)\in w_1^{\delta_1}...w_n^{\delta_n}\mathcal{D}'_{L^1}$ it suffices to show that the tempered distribution $w_1^{-\delta_1}...w_n^{-\delta_n}\mathcal{H}(T)$ is continuous in C_0^∞ with the topology of $\dot{B}$.

Let $\varphi\in C_0^\infty$, then

$$\begin{aligned}&\left|\left\langle w_1^{-\delta_1}...w_n^{-\delta_n}\mathcal{H}(T),\varphi\right\rangle_{\mathcal{S}',\mathcal{S}}\right|\\&=\left|(-1)^n\left\langle w_1^{-1}...w_n^{-1}T,w_1...w_n\mathcal{H}\left(w_1^{-\delta_1}...w_n^{-\delta_n}\varphi\right)\right\rangle_{\mathcal{D}'_{L^1},\dot{B}}\right|\\&\le\int_{\mathbb{R}^n}|f(x)|\left|\partial^\alpha\left[w_1...w_n\mathcal{H}\left(w_1^{-\delta_1}...w_n^{-\delta_n}\varphi\right)\right](x)\right|dx.\end{aligned}\tag{15}$$

Denote by $\psi=w_1^{-\delta_1}...w_n^{-\delta_n}\varphi\in C_0^\infty$. We need to estimate

$$\partial^\alpha\left[w_1...w_n\mathcal{H}(\psi)\right]=\sum_{\beta\le\alpha}C_{\alpha\beta}\partial^{\beta_1}w_1...\partial^{\beta_n}w_n\mathcal{H}\left(\partial^{\alpha-\beta}\psi\right).\tag{16}$$

We have that

$$\begin{aligned}\mathcal{H}(\psi)(x)&=\frac{1}{\pi^n}\lim_{\varepsilon_1\to0,...,\varepsilon_n\to0}\int_{|y_1|>\varepsilon_1}...\int_{|y_n|>\varepsilon_n}\frac{\psi(x_1-y_1,...,x_n-y_n)}{y_1...y_n}dy_n...dy_1\\&=\frac{1}{\pi^n}\lim_{\varepsilon_1\to0,...,\varepsilon_n\to0}\sum_{k=1}^{2^n}I_k,\end{aligned}$$

where $I_k=\int_{A_1}...\int_{A_n}$ and $A_j=\{\varepsilon_j<|y_j|<1\}$ or $A_j=\{1<|y_j|\}$, $j=1,...,n$.

Let us consider any of these integrals, say, for example

$$I=\int_{|y_1|>1}...\int_{|y_{n-1}|>1}\int_{\varepsilon_n<|y_n|<1}$$

(the other integrals can be analyzed in a similar way).

Thus

$$|I| \leq \int_{|y_1|>1} ... \int_{|y_{n-1}|>1} \int_{\varepsilon_n<|y_n|<1}$$
$$\times \left| \frac{\psi(x_1 - y_1, ..., x_{n-1} - y_{n-1}, x_n - y_n) - \psi(x_1 - y_1, ..., x_{n-1} - y_{n-1}, x_n)}{y_1 ... y_{n-1} y_n} \right| dy_n ... dy_1$$
$$\leq \int_{|y_1|>1} ... \int_{|y_{n-1}|>1} \int_{\varepsilon_n<|y_n|<1} \int_0^1 \frac{\left| \frac{\partial \psi}{\partial x_n}(x_1 - y_1, ..., x_{n-1} - y_{n-1}, x_n - t y_n) \right|}{|y_1 ... y_{n-1}|} dt dy_n ... dy_1.$$

Using the inequalities

$$\left(1 + x_j^2\right)^{1/2} \leq C_j \left(1 + (x_j - y_j)^2\right)^{-1/2} |y_j|$$

for $|y_j| > 1$, $j = 1, ..., n-1$, and

$$\left(1 + x_n^2\right)^{1/2} \leq C_n \left(1 + (x_n - t y_n)^2\right)^{-1/2}$$

if $0 \leq t \leq 1$ and $|y_n| < 1$, we obtain for η_j such that $1 < \eta_j < 1 + \delta_j$, $j = 1, ..., n$

$$w_1(x_1) ... w_n(x_n) |I|$$
$$\leq C \int_{|y_1|>1} ... \int_{|y_{n-1}|>1} \int_{\varepsilon_n<|y_n|<1} \int_0^1 w_1^{-1}(x_1 - y_1) ... w_{n-1}^{-1}(x_{n-1} - y_{n-1}) w_n^{-1}(x_n - t y_n)$$
$$\times \left| \frac{\partial \psi}{\partial x_n}(x_1 - y_1, ..., x_{n-1} - y_{n-1}, x_n - t y_n) \right| dt dy_n ... dy_1$$
$$= C \int_{|y_1|>1} w_1^{-\eta_1}(x_1 - y_1) ... \int_{|y_{n-1}|>1} w_{n-1}^{-\eta_{n-1}}(x_{n-1} - y_{n-1})$$
$$\times \int_{\varepsilon_n<|y_n|<1} \int_0^1 w_n^{-\eta_n}(x_n - t y_n) \left[w_1^{\eta_1 - 1}(x_1 - y_1) ... w_{n-1}^{\eta_{n-1} - 1}(x_{n-1} - y_{n-1}) w_n^{\eta_n - 1}(x_n - t y_n) \right.$$
$$\left. \times \left| \frac{\partial \psi}{\partial x_n}(x_1 - y_1, ..., x_{n-1} - y_{n-1}, x_n - t y_n) \right| \right] dt dy_n ... dy_1$$
$$\leq C \left\| \left(1 + \xi_1^2\right)^{(\eta_1 - 1)/2} ... \left(1 + \xi_n^2\right)^{(\eta_n - 1)/2} \frac{\partial \psi}{\partial \xi_n} \right\|_\infty \int_{|y_1|>1} w_1^{-\eta_1}(x_1 - y_1) dy_1$$
$$\times ... \int_{|y_{n-1}|>1} w_{n-1}^{-\eta_{n-1}}(x_{n-1} - y_{n-1}) dy_{n-1} \int_{\varepsilon_n<|y_n|<1} \int_0^1 w_n^{-\eta_n}(x_n - t y_n) dt dy_n$$
$$\leq C \left\| \left(1 + \xi_1^2\right)^{(\eta_1 - 1)/2} ... \left(1 + \xi_n^2\right)^{(\eta_n - 1)/2} \frac{\partial \psi}{\partial \xi_n} \right\|_\infty$$
$$\leq C \sup_{\xi \in \mathbb{R}^n} \left(1 + \xi_1^2\right)^{(\eta_1 - 1 - \delta_1)/2} ... \left(1 + \xi_n^2\right)^{(\eta_n - 1 - \delta_n)/2} \sum_{|\alpha| \leq 1} \|\partial^\alpha \varphi\|_\infty$$
$$\leq C \sum_{|\alpha| \leq 1} \|\partial^\alpha \varphi\|_\infty .$$

Thus, (15) can be estimated as

$$\int_{\mathbb{R}^n} |f(x)| \left| w_1...w_n \mathcal{H}\left(w_1^{-\delta_1}...w_n^{-\delta_n}\varphi\right)(x)\right| dx \le C \sum_{|\alpha|\le 1} \|\partial^\alpha \varphi\|_\infty \|f\|_1 .$$

The above computations show that

$$T \in L^1\left(w_1^{-1}...w_n^{-1}\right) \text{ implies } \mathcal{H}T \in w_1^{\delta_1}...w_n^{\delta_n}\mathcal{D}'_{L^1}. \tag{17}$$

Now, let us suppose that $|\alpha| \ge 0$.

We first notice that if $f \in L^1\left(w_1^{-1}...w_n^{-1}\right)$ then

$$\mathcal{H}(\partial^\alpha f) = \partial^\alpha \mathcal{H}(f). \tag{18}$$

The relation (18) can be easily proved just using (12), (13) and the fact that (18) is valid for every $\varphi \in \mathcal{S}$.

Therefore, given $T \in w_1...w_n\mathcal{D}'_{L^1}$, let us say $T = \partial^\alpha f$ with $f \in L^1\left(w_1^{-1}...w_n^{-1}\right)$, according to (17) we have that $\mathcal{H}(T) = \partial^\alpha \mathcal{H}(f) \in w_1^{\delta_1}...w_n^{\delta_n}\mathcal{D}'_{L^1}$ since this last space is closed under differentiation because

$$w_1^{\delta_1}...w_n^{\delta_n}\mathcal{D}'_{L^1} = \left\{ T \in \mathcal{S}' : T = \sum_{finite} \partial^\alpha f_\alpha,\ f_\alpha \in L^1\left(w_1^{-\delta_1}...w_n^{-\delta_n}\right)\right\}$$

(see [4]).

Finally, to show that $\mathcal{H}\left(w_1...w_n\mathcal{D}'_{L^1}\right)$ is not included in $\mathcal{D}'_{L^1}$, it suffices to notice that the distribution $T = \delta \in \mathcal{D}'_{L^1}$ and $\mathcal{H}(\delta) = \frac{1}{\pi^n} p.v. \frac{1}{x_1} \otimes ... \otimes p.v. \frac{1}{x_n} \notin \mathcal{D}'_{L^1}$ (see [2, Prop. 28]). ■

The following result is crucial.

Proposition 16 *Given $T \in w_1...w_n\mathcal{D}'_{L^1}$, the $\mathcal{S}'$-convolution $T * \mathcal{P}_{(y)}$ converges to T in $w_1...w_n\mathcal{D}'_{L^1}$ as $(y) \to (0)^+$.*

Proof. It will be enough to prove that for every $f \in L^1\left(w_1^{-1}...w_n^{-n}\right)$ we have $\mathcal{P}_{(y)} * f \to f$ in $L^1\left(w_1^{-1}...w_n^{-n}\right)$ as $(y) \to (0)^+$, since $L^1\left(w_1^{-1}...w_n^{-n}\right)$ is continuously included in $w_1...w_n\mathcal{D}'_{L^1}$ and the latter space is closed under differentiation.

Let us assume first that $f = \varphi_1(x_1)...\varphi_n(x_n)$ with $\varphi_j \in C_c(\mathbb{R})$ for $j = 1, ..., n$. Thus

$$\left\|\mathcal{P}_{(y)} * f - f\right\|_{L^1(\mathbb{R}^n)} \le \sum_{j=1}^n \left[\prod_{i\ne j} \|\varphi_i\|_\infty\right] \left\|P_{y_j} * \varphi_j - \varphi_j\right\|_{L^1(\mathbb{R})} \to 0$$

as $y_1, ..., y_n \to 0^+$ since $\varphi_j \in L^1(\mathbb{R})$, $j = 1, ..., n$.

As a consequence, when f is any function in $C_c(\mathbb{R}) \otimes ... \otimes C_c(\mathbb{R})$ we can easily show that

$$\left\|\mathcal{P}_{(y)} * f - f\right\|_{L^1(\mathbb{R}^n)} \to 0 \text{ as } (y) \to (0)^+$$

and since $L^1(\mathbb{R}^n)$ is continuously included in $L^1\left(w_1^{-1}...w_n^{-n}\right)$, the assertion is proved for this kind of functions.

Now, let us consider any function $f \in L^1\left(w_1^{-1}...w_n^{-n}\right)$ and let $\varepsilon > 0$. Since $C_c(\mathbb{R}) \otimes ... \otimes C_c(\mathbb{R})$ is dense in $L^1\left(w_1^{-1}...w_n^{-n}\right)$, we can find a function $g \in C_c(\mathbb{R}) \otimes ... \otimes C_c(\mathbb{R})$ such that

$$\|f - g\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} < \varepsilon.$$

Hence

$$\begin{aligned}
\left\|\mathcal{P}_{(y)} * f - f\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} &\leq \left\|\mathcal{P}_{(y)} * (f-g)\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} \\
&\quad + \left\|\mathcal{P}_{(y)} * g - g\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} + \varepsilon \\
&\leq \int_{\mathbb{R}^n} |f(t) - g(t)| \left[\prod_{j=1}^{n} \int_{-\infty}^{\infty} P_{y_j}(x_j - t_j) \frac{dx_j}{\left(1 + x_j^2\right)^{1/2}}\right] dt \\
&\quad + \left\|\mathcal{P}_{(y)} * g - g\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} + \varepsilon \\
&= \int_{\mathbb{R}^n} |f(t) - g(t)| \left[\prod_{j=1}^{n} \left(P_{y_j} * w_j^{-1}\right)(t_j)\right] dt \\
&\quad + \left\|\mathcal{P}_{(y)} * g - g\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} + \varepsilon.
\end{aligned}$$

Since $|f(t) - g(t)| \left[\prod_{j=1}^{n} \left(P_{y_j} * w_j^{-1}\right)(t_j)\right] \to |f(t) - g(t)| \prod_{j=1}^{n} w_j^{-1}(t_j)$ as $(y) \to (0)^+$ and $f - g \in L^1\left(w_1^{-1}...w_n^{-n}\right)$, if we were able to show that the family $|f(t) - g(t)| \left[\prod_{j=1}^{n} \left(P_{y_j} * w_j^{-1}\right)(t_j)\right]$, $0 < y_j < 1$, $j = 1, ..., n$, is dominated by an integrable function, we would have for $y_1, ..., y_n$ small enough that

$$\left\|\mathcal{P}_{(y)} * f - f\right\|_{L^1\left(w_1^{-1}...w_n^{-n}\right)} < 4\varepsilon$$

and the Proposition would be proved.

To show that $|f(t) - g(t)| \left[\prod_{j=1}^{n} \left(P_{y_j} * w_j^{-1}\right)(t_j)\right]$ is controlled by an integrable function, we proceed as follows. Fix $j = 1, ..., n$ and write $d\mu_j(x_j) = P_{y_j}(x_j)\, dx_j$, $\psi_j(u) = u^{1/2}$, $u \geq 0$. We observe that $\mu_j(\mathbb{R}) = 1$ and ψ_j is a concave function. Now, using Jensen's inequality we obtain for $0 < y_j < 1$

$$\begin{aligned}
\left(P_{y_j} * w_j^{-1}\right)(t_j) &= \int_{-\infty}^{\infty} \frac{1}{\left(1+(t_j-x_j)^2\right)^{1/2}} d\mu_j(x_j) \\
&= \int_{-\infty}^{\infty} \psi_j \left(\frac{1}{1+(t_j-x_j)^2}\right) d\mu_j(x_j) \\
&\leq \psi_j \left(\int_{-\infty}^{\infty} \frac{1}{1+(t_j-x_j)^2} d\mu_j(x_j)\right) \\
&= C\left(\int_{-\infty}^{\infty} P_1(t_j-x_j) P_{y_j}(x_j)\, dx_j\right)^{1/2} \\
&= C\left(P_{1+y_j}(t_j)\right)^{1/2} \\
&\leq C\frac{1}{\left(1+t_j^2\right)^{1/2}}.
\end{aligned}$$

Thus,

$$|f(t)-g(t)| \left[\prod_{j=1}^{n} \left(P_{y_j} * w_j^{-1}\right)(t_j)\right] \leq C\,|f(t)-g(t)| \prod_{j=1}^{n} \frac{1}{\left(1+t_j^2\right)^{1/2}} \tag{19}$$

and the right hand side of (19) is an integrable function on $\mathbb{R}^n$.

This completes the proof. ∎

Lemma 17 *$\mathcal{H}(T) * \mathcal{P}_{(y)} = T * \mathcal{Q}_{(y)}$ for every $T \in w_1...w_n\mathcal{D}'_{L^1}$ and $(y) > (0)$.*

Proof. Let $\varphi \in \mathcal{S}$, then taking into account that $\mathcal{H}(T) \in w_1...w_n\mathcal{D}'_{L^1}$ and $w_1...w_n\left(\mathcal{P}_{(y)} * \varphi\right) \in B_c$ we have that

$$\begin{aligned}
\left(\mathcal{H}(T) * \mathcal{P}_{(y)}, \varphi\right)_{\mathcal{S}',\mathcal{S}} &= \left(\mathcal{H}(T)\left(\check{\mathcal{P}}_{(y)} * \varphi\right), 1\right)_{\mathcal{D}'_{L^1}, B_c} \\
&= \left(\mathcal{H}(T), \mathcal{P}_{(y)} * \varphi\right)_{w_1...w_n\mathcal{D}'_{L^1}, w_1^{-1}...w_n^{-1}B_c}.
\end{aligned} \tag{20}$$

On the other hand, since $w_1...w_n\left(\mathcal{Q}_{(y)} * \varphi\right) \in B_c$

$$\begin{aligned}
\left(T * \mathcal{Q}_{(y)}, \varphi\right)_{\mathcal{S}',\mathcal{S}} &= \left(T\left(\check{\mathcal{Q}}_{(y)} * \varphi\right), 1\right)_{\mathcal{D}'_{L^1}, B_c} \\
&= \left(T, \mathcal{Q}_{(y)} * \varphi\right)_{w_1...w_n\mathcal{D}'_{L^1}, w_1^{-1}...w_n^{-1}B_c} \\
&= \left(T, (-1)^n \mathcal{H}\left(\mathcal{P}_{(y)}\right) * \varphi\right)_{w_1...w_n\mathcal{D}'_{L^1}, w_1^{-1}...w_n^{-1}B_c} \\
&= \left(T, (-1)^n \left(p.v.\frac{1}{x_1} \otimes ... \otimes p.v.\frac{1}{x_n}\right) * \mathcal{P}_{(y)} * \varphi\right)_{w_1...w_n\mathcal{D}'_{L^1}, w_1^{-1}...w_n^{-1}B_c} \\
&= \left(\mathcal{H}(T), \mathcal{P}_{(y)} * \varphi\right)_{w_1...w_n\mathcal{D}'_{L^1}, w_1^{-1}...w_n^{-1}B_c}.
\end{aligned} \tag{21}$$

From (20) and (21) we obtain that $\mathcal{H}(T) * \mathcal{P}_{(y)} = T * \mathcal{Q}_{(y)}$. ∎

As an immediate consequence of the previous results we can obtain:

Theorem 18 *Given $T \in w_1...w_n\mathcal{D}'_{L^1}$, the $\mathcal{S}'$-convolution $T * \mathcal{Q}_{(y)}$ converges to $\mathcal{H}(T)$ in $w_1...w_n\mathcal{D}'_{L^1}$ as $(y) \to (0)^+$.*

Proof. Since $\mathcal{H}(T) \in w_1...w_n\mathcal{D}'_{L^1}$, according to Proposition 16

$$\mathcal{H}(T) * \mathcal{P}_{(y)} \to \mathcal{H}(T)$$

in $w_1...w_n\mathcal{D}'_{L^1}$ as $(y) \to (0)^+$. Now, by Lemma 17 we get the desired conclusion. ■

References

[1] Alvarez, J. and Carton-Lebrun, C., 1999, Optimal spaces for the $\mathcal{S}'$-convolution with the Marcel Riesz kernels and the n-dimensional Hilbert kernel, In: W.O. Bray and Č.V. Stanojević (Ed.) *Analysis of Divergence: Control and Management of Divergent Processes* (Boston: Birkhäuser), 233-248.

[2] Alvarez, J. and Guzmán-Partida, M., 2002, The $\mathcal{S}'$-convolution with singular kernels in the euclidean case and the product domain case. *Journal of Mathematical Analysis and Applications*, **270**, 405-434.

[3] Alvarez, J., Guzmán-Partida, M. and Skórnik, U., 2003, $\mathcal{S}'$-convolvability with the Poisson kernel in the Euclidean case and the product domain case, *Studia Mathematica*, **156** (2), 143-163.

[4] Alvarez, J., Guzmán-Partida, M. and Pérez-Esteva, S., Harmonic extensions of distributions, To appear in *Mathematische Nachrichten*.

[5] Alvarez, J., Guzmán-Partida, M. and Pérez-Esteva, S., 2006, N-harmonic extensions of weighted integrable distributions, *Journal of Function Spaces and Applications*, **4** (3), 305-327.

[6] Barros-Neto, J., 1973, *An introduction to the theory of distributions* (New York: Marcel Dekker).

[7] Dierolf, P. and Voigt, J., 1978, Convolution and $\mathcal{S}'$-convolution of distributions, *Collectanea Mathematica*, **29**, 185-196.

[8] Hirata, Y. and Ogata, H., 1958, On the exchange formula for distributions, *Journal of Sciences of the Hiroshima University*, Series A, **22**, 147-152.

[9] Pandey, J.N. and Singh, O.P., 1994, Characterization of functions with Fourier transform supported on orthants, *Journal of Mathematical Analysis and Applications*, **185**, 438-463.

[10] Schwartz, L., 1962, Causalité et analyticité, *Anais da Academia Brasileira de Ciências*, **34**, 13-21.

[11] Schwartz, L., 1966, *Théorie des distributions* (Paris: Hermann).

[12] Shiraishi, R., 1959, On the definition of convolutions for distributions, *Journal of Sciences of the Hiroshima University*, Series A, **23**, 19-32.

[13] Singh, O.P. and Pandey, J.N., 1990, The n-dimensional Hilbert transform of distributions, its inversion and applications, *Canadian Journal of Mathematics*, **42** (2), 239-258.